Midterm

MTH 616, Semester 2, 2022-23

Duration: 2 hours

Maximum Points: 50

Directions: Please ensure that you sufficiently explain and justify all intermediate arguments leading to any conclusions you may draw. Each statement (or argument) in your solution must be clearly presented, and must be devoid of any logical gaps or inconsistencies.

1. Let A, B be a pair of subspaces of a space X such that $X = A^\circ \cup B^\circ$. Then there exists a long exact sequence of homology groups known as the *Mayer-Vietoris sequence* given by

$$\ldots \to H_n(A\cap B) \xrightarrow{\phi} H_n(A)\oplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A\cap B) \to \ldots \to H_0(X) \to 0,$$

which is associated with the short exact sequence of chain complexes

$$0 \to C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A + B) \to 0,$$

where $\phi(x) = (x, -x)$, $\psi(x, y) = x + y$, and ∂ is induced by the usual boundary map $\partial : C_n(X) \to C_{n-1}(X)$. The Mayer-Vietoris sequence for homology can be viewed as an analog of the Seifert van Kampen Theorem for fundamental groups.

   (a) Using the Mayer-Vietoris sequence, establish that $\widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X)$ for all n.

   (b) Using the Mayer-Vietoris sequence, compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus. [5+10]

2. Let X be space and A a subspace of X such that $r : X \to A$ is a retraction.

   (a) Show that $H_n(X) \cong H_n(A) \oplus H_n(X, A)$.

   (b) Using (a), compute the homology group of closed orientable surface of genus g with one puncture. [5+10]

3. The *Euler characteristic* $\chi(X)$ of a Δ-complex X of dimension n is defined by

$$\chi(X) := \sum_{i=0}^{n} (-1)^i \beta_i,$$

where β_i is the number of i-simplices in X.

   (a) Show that $\chi(X) = \displaystyle\sum_{i=0}^{n} (-1)^i \mathrm{Rank}(H_i^\Delta(X))$.

   (b) Let N_g be the closed nonorientable surface of genus g (or g crosscaps.) Show that $\chi(N_g) = 2 - g$. [10+10]